J. Fluid Mech. (1973), *vol.* 60, *part* 1, *pp.* 81–96
Printed in Great Britain

Pressure drop due to the motion of a sphere near the wall bounding a Poiseuille flow

By PETER M. BUNGAY† AND HOWARD BRENNER

Department of Chemical Engineering and Biotechnology Program,
Carnegie–Mellon University, Pittsburgh, Pennsylvania 15213

(Received 16 January 1973)

An expression is derived for the (low Reynolds number) additional pressure drop created by a relatively small sphere moving near the wall of a circular tube through which there is a Poiseuille flow. Two specific applications are examined: (i) the sedimentation of a homogeneous non-neutrally buoyant sphere in a quiescent fluid; and (ii) the motion of a neutrally buoyant sphere. In the latter case a pronounced increase in the additional pressure drop is predicted when the separation between the sphere and the tube wall is reduced to zero.

This analysis, which includes the behaviour for a sphere in contact with the tube wall, supplements previous 'method of reflexions' treatments valid only when the distance from the sphere centre to the wall is large compared with the sphere radius.

1. Introduction

One of the fundamental theoretical problems in the slow viscous duct flow of dilute, rigid-sphere suspensions is the description of the behaviour of a single sphere immersed in a fluid bounded by a circular cylindrical tube. The case of greatest interest in applications is that in which the sphere radius is small compared with that of the cylinder. This problem can be viewed as one of determining the hydrodynamic influence of the tube wall upon the suspended sphere. The magnitude of this interaction effect depends *inter alia* upon the size and lateral position of the sphere relative to the tube wall.

Prior analyses (based upon 'method of reflexions' or equivalent computational schemes) have been concerned exclusively with the case where the distance of the sphere from the tube wall is large compared with the sphere radius. The present work is devoted to a treatment of the opposite case, where the sphere is situated very near to the wall. It will be demonstrated that very considerable alterations can occur in the translational and rotational particle velocities, and in the additional pressure drop (above the purely Poiseuillian portion of the pressure drop) when a small sphere is in close proximity to the wall, in contrast to the case where it is relatively distinct. The consequences of this hydrodynamic wall effect are pertinent to a proper understanding of suspension sedimentation and rheology.

† Present address: University Medical Clinic, Montreal General Hospital, 1650 Cedar Avenue, Montreal 109.

Consider a homogeneous sphere (radius $= a$, density $= \rho_s$) immersed in an incompressible Newtonian fluid (viscosity $= \mu$, density $= \rho$) confined within a long circular cylindrical tube of radius R_0. When the sphere is not neutrally buoyant ($\rho_s \neq \rho$) the tube will be assumed vertical, with its symmetry axis parallel to gravity. As in figure 1, let h denote the radial distance from the sphere centre to the nearest point on the tube wall.

All prior theoretical treatments of flow past an eccentrically positioned sphere have been concerned with the limit in which the sphere is small compared with its distance from the wall, i.e. $a \ll h \leqslant R_0$. As shown by Cox & Brenner (1967) the slow viscous motion in this limit can be analysed via regular perturbation expansion procedures. Alternatively, equivalent asymptotic expansions may be derived by application of the 'method of reflexions'. This latter technique was applied by Brenner & Happel (1958) to Poiseuille flow past a sphere translating, without rotation, parallel to the tube axis. Higher order terms in a/R_0 were subsequently evaluated by Greenstein & Happel (1968, 1970), who also included the effect of sphere rotation. The results of Greenstein & Happel (1970) provide an asymptotic expression for the additional pressure drop in the important special case in which sedimentation and other buoyancy effects are negligible. Concurrently, Brenner (1970) demonstrated that the same neutrally buoyant pressure drop terms could be obtained without the detailed knowledge of the boundary-value solutions required by the method of reflexions. When the sphere is constrained to translate (without rotation) along the tube axis ($h = R_0$) the fluid motion is axisymmetric, and hence easily susceptible to higher order analysis (Haberman & Sayre 1958; Hochmuth & Sutera 1970; Wang & Skalak 1969).

Because of convergence difficulties the method of reflexions, as well as equivalent expansion procedures, cannot be used to examine the hydrodynamic particle–wall interaction experienced by an eccentrically positioned sphere in the case of either (a) a small sphere in close proximity to the tube wall, $a \leqslant h \ll R_0$; or (b) a closely fitting sphere, $a \leqslant h \leqslant R_0$, $a = O(R_0)$. The proper analysis of case (b), which necessitates employing singular perturbation techniques, is treated in a companion paper (Bungay & Brenner 1973*b*; see also Bungay 1970). In the present study an alternative regular perturbation procedure is outlined for case (a) to complement the method of reflexions.

Before confining attention to this case we shall derive reciprocity relationships between the additional pressure drop and the hydrodynamic force and torque acting on a sphere. These apply without restriction as to sphere/tube size or radial position in the tube.

2. Sphere of arbitrary size and position

Formulation of the problem

In figure 1 a Cartesian co-ordinate system (x, y, z) is defined with origin at the centre o of the sphere. The unit vector $\hat{\mathbf{z}}$ of the co-ordinate system is parallel to the tube axis. Both the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ unit vectors lie in the meridian plane containing

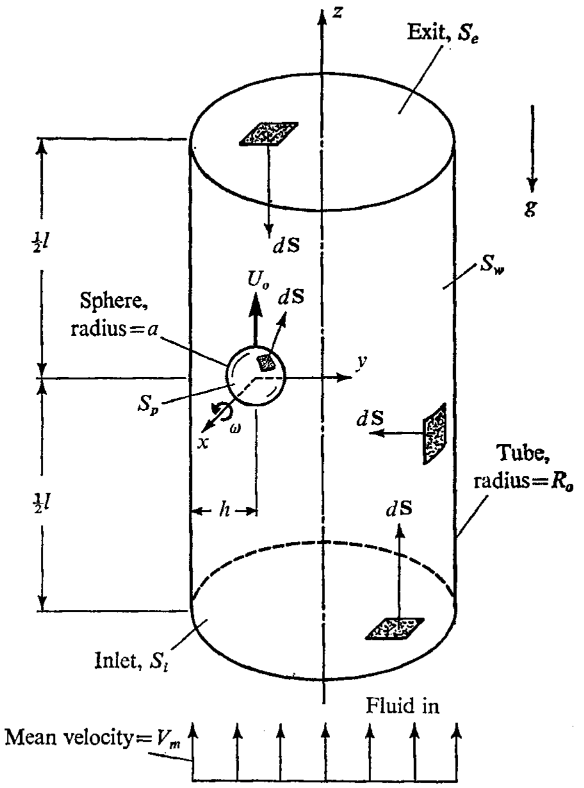

FIGURE 1. Eccentrically positioned sphere in a vertical cylindrical tube.

the tube axis and passing through the sphere centre. The cross-sectional plane perpendicular to the tube axis and passing through the point o constitutes a second plane of geometrical reflexion symmetry.

The suspending fluid flows upward through the vertical tube at a steady superficial mean velocity V_m. Motion of the sphere is prescribed by the translational velocity $\mathbf{U}_0$ of its centre and its angular velocity $\boldsymbol{\omega}$. Accordingly, in consequence of the no-slip condition, the local fluid motion satisfies the boundary condition

$$\mathbf{v} = \mathbf{U}_0 + \boldsymbol{\omega} \times \mathbf{r} \quad \text{on the sphere,} \tag{2.1}$$

in addition to which

$$\mathbf{v} = \mathbf{0} \quad \text{on the tube wall,} \tag{2.2}$$

where $\mathbf{r}$ is the position vector relative to an origin at the sphere centre. The three characteristic velocities $\mathbf{U}_0$, $\boldsymbol{\omega}$ and V_m may be tentatively regarded as independent quantities. In consequence of these motions the particle is acted upon by a hydrodynamic force $\mathbf{F}$ and a hydrodynamic torque $\mathbf{T}_0$ about the sphere centre.

All relevant particle Reynolds numbers are assumed sufficiently small compared with unity to justify neglect of the inertial terms in the Navier–Stokes equations. After the hydrostatic term has been incorporated in the definition of the dynamic pressure field p, the dynamical and kinematical equations governing the fluid motion thereby reduce to the Stokes equations,

$$\mu \nabla^2 \mathbf{v} = \nabla p \tag{2.3}$$

and

$$\nabla . \mathbf{v} = 0. \tag{2.4}$$

As demonstrated by Cox & Mason (1971) for the case where gravity represents the only external force, the sphere cannot experience any lateral movement. Rather, the sphere, if translating, necessarily travels parallel to the tube axis. Hence, we may write

$$\mathbf{U}_o = U_o\hat{\mathbf{z}}, \quad \mathbf{F} = F\hat{\mathbf{z}}. \tag{2.5a, b}$$

By symmetry arguments based upon the sphere–tube geometry, the translational motion (2.5a) can give rise to a hydrodynamic torque possessing a component only in the x direction. If external torques other than those acting parallel to $\hat{\mathbf{x}}$ are supposed absent in general, then we may also prescribe that

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}, \quad \mathbf{T}_o = T_o\hat{\mathbf{x}}. \tag{2.6a, b}$$

The disturbance to the Poiseuille flow caused by the presence of the sphere decays exponentially with distance from the particle (Sonshine, Cox & Brenner 1966). Consider a hypothetical 'inlet' plane S_i located at a distance of $\frac{1}{2}l$ units upstream of the sphere and an 'exit' plane S_e at a similar distance downstream. For sufficiently large l the local velocity field is Poiseuillian:

$$\mathbf{v}(S_i), \mathbf{v}(S_e) \to u\hat{\mathbf{z}} \quad \text{as} \quad l \to \infty, \tag{2.7}$$

with

$$u = 2V_m[1-(R/R_o)^2], \tag{2.8}$$

in which R is the radial distance from the tube axis. The dynamic pressure across each of these two distant planes is uniform, and consequently can be written as

$$\left.\begin{aligned} p(S_i) &\to \overset{*}{p}(S_i) + \tfrac{1}{2}\Delta P^+ \\ p(S_e) &\to \overset{*}{p}(S_e) - \tfrac{1}{2}\Delta P^+ \end{aligned}\right\} \quad \text{as} \quad l \to \infty, \tag{2.9}$$

where $\overset{*}{p} = -8\mu V_m z/R_o^2 + C$ is the Poiseuille pressure distribution for flow at mean velocity V_m in the absence of the sphere, C being an arbitrary constant. Let

$$\Delta P = p(S_i) - p(S_e) \tag{2.10}$$

and

$$\Delta\overset{*}{P} = \overset{*}{p}(S_i) - \overset{*}{p}(S_e), \tag{2.11}$$

respectively, denote the pressure drops in the presence and absence of the particle from the Poiseuille flow. Then,

$$\Delta P^+ = \lim_{l\to\infty} (\Delta P - \Delta\overset{*}{P}) \tag{2.12}$$

represents the additional pressure drop due to the presence of the sphere in the flow.

In view of the linearity of the Stokes equations and of the boundary conditions, the hydrodynamic force, torque and additional pressure drop are necessarily linear functions of each of the three characteristic velocities. Thus, we may write

$$\begin{pmatrix} F \\ T_o \\ \Delta P^+ A \end{pmatrix} = -\mu \begin{pmatrix} K^t & K^r & K^s \\ L^t & L^r & L^s \\ P^t & P^r & P^s \end{pmatrix} \begin{pmatrix} U_o \\ \omega \\ -V_m \end{pmatrix}, \tag{2.13}$$

in which $A = \pi R_o^2$ is the tube cross-sectional area, $\Delta P^+ A$ then being the additional pressure drop force. The matrix composed of the nine 'intrinsic hydrodynamic resistance' coefficients is a purely geometrical function, dependent only upon the sphere and tube radii, and upon the relative radial position of the sphere in the tube.

The scalar elements of the hydrodynamic resistance matrix are all positive, with the exception of K^r and L^t. In the perturbed motion, representing the difference between the flows in the presence and absence of the particle, the rate $\dot{W}$ at which work is being done by the stresses acting over the surfaces bounding the fluid is $\dot{W} = -(FU_o + T_o\omega - \Delta P^+ A V_m)$. In creeping flow this quantity is identical to the rate of mechanical energy dissipation $\dot{E}$ in the fluid bounded by the surfaces S_i and S_e, the tube wall S_w and the sphere surface S_p. Since this dissipation rate is necessarily non-negative it follows that

$$\dot{E} = -(U_o, \omega, -V_m)\begin{pmatrix} F \\ T_o \\ \Delta P^+ A \end{pmatrix} \geqslant 0, \tag{2.14}$$

whence the resistance matrix is a positive-definite form.

The resistance matrix also enjoys the property of being symmetric, corresponding to the three reciprocity relations

$$K^r = L^t, \quad P^t = K^s, \quad P^r = L^s. \qquad (2.15), (2.16), (2.17)$$

The first of these is proved by Brenner (1964). Demonstrations of the validity of the remaining two Onsager-like reciprocal relations are furnished in the following paragraphs. Equations (2.13)–(2.17) apply even for non-circular cylindrical ducts.

Reciprocal relations

Derivations of (2.16) and (2.17) may be formulated via the Lorentz reciprocal theorem using techniques previously applied in the proof of (2.15). Let $\mathbf{v}'$ and $\mathbf{v}''$ be any pair of velocity fields satisfying the Stokes equations, and let $\boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$ denote the corresponding pressure tensors: $\boldsymbol{\pi} = -\mathbf{I}p + \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger]$. According to the reciprocal theorem,

$$\oint_S d\mathbf{S}.\boldsymbol{\pi}'.\mathbf{v}'' = \oint_S d\mathbf{S}.\boldsymbol{\pi}''.\mathbf{v}'. \tag{2.18}$$

Here, S is an arbitrary closed surface bounding a continuous volume V of the homogeneous fluid, and $d\mathbf{S}$ is a differential element of surface area directed into the fluid volume V. For present purposes, S is chosen to consist of the planes S_e and S_i, the portion S_w of the inside wall of the tube between these two planes, and the surface S_p of the sphere

$$S = S_e + S_i + S_p + S_w. \tag{2.19}$$

To deduce the cross-relationship (2.16), let the primed motion denote flow past a sphere which is stationary with respect to the tube. Thus, at the rigid surfaces,

$$\mathbf{v}' = \mathbf{0} \quad \text{on} \quad S \quad \text{and} \quad S_w, \qquad (2.20a, b)$$

while up- and downstream of the sphere,

$$\mathbf{v}' \to u\hat{\mathbf{z}} \quad \text{on} \quad S_e \quad \text{and} \quad S_i. \tag{2.20c}$$

Choice of the double-primed flow to be that resulting from the sphere translating, without rotation, through an otherwise quiescent medium ($V_m = 0$) leads to the boundary conditions

$$\mathbf{v}'' = U_0\hat{\mathbf{z}} \quad \text{on} \quad S_p, \quad \mathbf{v}'' = \mathbf{0} \quad \text{on} \quad S_w \tag{2.21a,b}$$

and

$$\mathbf{v}'' \to \mathbf{0} \quad \text{on} \quad S_e \quad \text{and} \quad S_i. \tag{2.21c}$$

From the last of these conditions it follows that $\boldsymbol{\pi}'' \to \mathbf{I}p''$, with p'' being uniform on both S_e and S_i. This limit condition, together with (2.20a,b,c), when substituted into the right-hand side of (2.18), along with $d\mathbf{S} = +\hat{\mathbf{z}}\,dS$ on S_i and

$$d\mathbf{S} = -\hat{\mathbf{z}}\,dS \quad \text{on} \quad S_e,$$

yields

$$\oint_S d\mathbf{S}\,.\,\boldsymbol{\pi}''\,.\,\mathbf{v}' = [p''(S_e)-p''(S_i)]\int_{S_i} u\,dS \to \mu P^t U_0 V_m. \tag{2.22}$$

In arriving at the above, (2.12) and (2.13) have been used along with

$$V_m = A^{-1}\int_{S_i} u\,dS \equiv A^{-1}\int_{S_e} u\,dS.$$

Application of (2.5b) and (2.13), and use of the integral relation

$$\mathbf{F} = \int_{S_p} d\mathbf{S}\,.\,\boldsymbol{\pi}$$

to evaluate the hydrodynamic force exerted on the sphere in the single-primed case, leads to the relation

$$\int_{S_p} d\mathbf{S}\,.\,\boldsymbol{\pi}'\,.\,\hat{\mathbf{z}} = \mu K^s V_m.$$

This expression, together with boundary conditions (2.21a,b,c), simplifies the left-hand side of (2.18) to the form

$$\oint_S d\mathbf{S}\,.\,\boldsymbol{\pi}'\,.\,\mathbf{v}'' = \mu K^s U_0 V_m. \tag{2.23}$$

Identity (2.16) is established by equating (2.22) and (2.23). The second identity, (2.17), is proved in the same manner upon choosing the double-primed flow to be that arising solely from rotation of the sphere ($U_0 = V_m = 0$).

The reciprocity relation analogous to (2.16) for the motion of a spherical fluid droplet within a duct of arbitrary cross-section was derived by Bungay & Brenner (1973a).

3. Small sphere near the tube wall

Perturbation expansion

Consider the case of a sphere whose relative size and lateral position in the tube satisfy the dual constraints

$$\lambda \ll a/h \leqslant 1, \tag{3.1}$$

in which

$$\lambda = a/R_0. \tag{3.2}$$

Expressed in the Cartesian co-ordinates of figure 1, the equation of the tube wall surface is $x^2+(R_o-h-y)^2 = R_o^2$. Use of the dimensionless co-ordinates

$$\hat{x} = x/a, \quad \hat{y} = y/a, \quad \hat{z} = z/a \tag{3.3}$$

permits the equation describing the wall to be expressed as a series expansion in powers of λ:

$$\hat{y} = -h/a+(\tfrac{1}{2}\hat{x}^2)\,\lambda+O(\lambda^2), \tag{3.4}$$

for $|\hat{x}| \ll \lambda^{-1}$. This expansion shows that, to terms of zero order in λ, the tube wall may be replaced by the plane surface $\hat{y} = -h/a$.

The asymptotic expression (3.4) for the surface defining the cylindrical boundary suggests that the velocity and pressure fields possess the following regular perturbation expansions:

$$\mathbf{v} = V_*[\mathbf{v}_0+\lambda\mathbf{v}_1+\ldots] \tag{3.5}$$

and

$$p = (\mu V_*/a)\,[p_0+\lambda p_1+\ldots], \tag{3.6}$$

in which V_* is a characteristic reference velocity to be determined. The local field pairs $(\mathbf{v}_0, p_0)$, $(\mathbf{v}_1, p_1), \ldots,$ depend upon $\hat{\mathbf{r}} = \mathbf{r}/a$ and the parameter a/h, but are independent of λ for $\lambda \ll a/h$. Upon substitution of the expansions (3.5) and (3.6) into the governing equations of § 2, the zero-order fields $(\mathbf{v}_0, p_0)$ are found to satisfy the following differential equations and boundary conditions:

$$\hat{\nabla}^2\mathbf{v}_0 = \hat{\nabla}p_0, \quad \hat{\nabla}\cdot\mathbf{v}_0 = 0, \tag{3.7a,b}$$

$$\mathbf{v}_0 = \hat{U}_o\hat{\mathbf{z}}+\hat{\omega}(\hat{\mathbf{x}}\times\hat{\mathbf{r}}) \quad \text{on the sphere} \quad |\hat{\mathbf{r}}| = 1, \tag{3.8}$$

$$\mathbf{v}_0 = \mathbf{0} \quad \text{on the plane} \quad \hat{y} = -h/a \tag{3.9}$$

and

$$\mathbf{v}_0 \to \mathbf{u}_0, \quad p_0 \to 0 \quad \text{as} \quad |\hat{\mathbf{r}}| \to \infty, \tag{3.10}$$

in which

$$\hat{U}_o = U_o/V_*, \quad \hat{\omega} = \omega a/V_* \tag{3.11a,b}$$

and

$$\hat{\nabla} = a\nabla. \tag{3.12}$$

The zero-order undisturbed velocity field $\mathbf{u}_0$ far from the sphere can be found by expressing the Poiseuille velocity distribution as a polynomial in λ of the form

$$\mathbf{u} = V_*[\mathbf{u}_0+\lambda\mathbf{u}_1+\ldots]. \tag{3.13}$$

A concise formulation of the zero-order field found in this way is

$$\mathbf{u}_0 = \hat{\mathbf{u}}_0+\hat{\mathbf{r}}\,.\,(\hat{\nabla}\hat{\mathbf{u}})_o, \tag{3.14}$$

in which

$$\hat{\mathbf{u}}_o = \hat{\mathbf{z}} \tag{3.15}$$

and

$$(\hat{\nabla}\hat{\mathbf{u}})_o = (a/h)\,\hat{\mathbf{y}}\hat{\mathbf{z}}. \tag{3.16}$$

This recasting of the Poiseuille field also shows that the reference velocity should be chosen as

$$V_* = 4V_m h/R_o. \tag{3.17}$$

Equation (3.14) constitutes a simple shear flow whose dimensional rate of shear $4V_m/R_o$ is the local Poiseuille shear rate at the tube wall. Equations (3.7)–(3.10) and (3.14) reveal that, to terms of zero order in λ, the original tube flow problem may be approximated by the more tractable problem of simple shear flow round a sphere in the neighbourhood of a plane wall.

Expansions (3.5) and (3.6) induce expansions of the hydrodynamic force and torque of the forms

$$F = 6\pi\mu V_* a[F_0 + \lambda F_1 + \ldots] \tag{3.18}$$

and

$$T_o = 8\pi\mu V_* a^2[(T_o)_0 + \lambda(T_o)_1 + \ldots]. \tag{3.19}$$

The zero-order contributions expressed in terms of dimensionless zero-order force and torque resistance coefficients are

$$F_0 = K_0^s - K_0^t \hat{U}_o - K_0^r \hat{\omega} \tag{3.20}$$

and

$$(T_o)_0 = L_0^s - L_0^t \hat{U}_o - L_0^r \hat{\omega}. \tag{3.21}$$

There exist no zero-order contributions to the pressure drop coefficients in (2.13). This is to be expected since no pressure drop can arise from the semi-infinite flow described by (3.7)–(3.10). However, the zero-order flow fields do give rise to pressure drops of second order in λ. Through application of the reciprocal theorem in a derivation similar to those of § 2, Brenner (1970) has shown that the additional pressure drop for a rigid particle in creeping duct flow can be calculated from the expression

$$\Delta P^+ V_m A = \int_{S_p} d\mathbf{S}.\boldsymbol{\pi}.\mathbf{u}. \tag{3.22}$$

The pressure tensor expanded in powers of λ is

$$\boldsymbol{\pi} = (\mu V_*/a)[\boldsymbol{\pi}_0 + \lambda\boldsymbol{\pi}_1 + \ldots]. \tag{3.23}$$

Substituting expansions (3.13) and (3.23) into (3.22), along with use of $d\mathbf{S} = a^2 d\hat{\mathbf{S}}$, yields

$$\Delta P^+ = (\mu V_*/R_o)[\lambda^2 \Delta P_2^+ + \lambda^3 \Delta P_3^+ + \ldots], \tag{3.24}$$

where, for the leading coefficient,

$$\Delta P_2^+ = \frac{4}{\pi}\left(\frac{h}{a}\right)\int_{S_p} d\hat{\mathbf{S}}.\boldsymbol{\pi}_0.\mathbf{u}_0, \tag{3.25}$$

in which $\boldsymbol{\pi}_0 = -\mathbf{I}p_0 + \hat{\nabla}\mathbf{v}_0 + (\hat{\nabla}\mathbf{v}_0)^\dagger$ is the dimensionless pressure tensor derived from the solution $(\mathbf{v}_0, p_0)$ of (3.7)–(3.10). This coefficient may be expressed linearly in terms of dimensionless second-order resistance coefficients as

$$\Delta P_2^+ = P_2^s - P_2^t \hat{U}_o - P_2^r \hat{\omega}. \tag{3.26}$$

The dimensionless reciprocity relations corresponding to (2.15)–(2.17) for the linear shear flow problem are then

$$K_0^r = \tfrac{4}{3} L_0^t, \tag{3.27}$$

$$P_2^t = 24(h/a)K_0^s, \quad P_2^r = 32(h/a)L_0^s. \qquad (3.28), (3.29)$$

These resistance coefficients are dependent only on a/h, being independent of λ. Their variation with the parameter a/h is discussed in the following subsection.

Evaluation of resistance coefficients

The motion of a sphere in the neighbourhood of a plane wall has been studied extensively. Goldman, Cox & Brenner (1967*a, b*) tabulate numerical values of the force and torque resistance coefficients over the full range of the parameter,

α	a/h	P_2^r	P_2^s	P_2^t
∞	0	16†	∞†	∞†
3·0	0·09933	15·9970	257·859	255·817
2·0	0·26580	15·9538	110·498	105·382
1·5	0·42510	15·8416	79·864	72·151
1·0	0·64805	15·5870	64·273	53·297
0·5	0·88684	15·2598	57·921	43·731
0·3	0·95666	15·1630	56·944	41·849
0·1	0·99502	15·1107	56·516	40·929
0·05	0·99875	15·1056	56·477	40·844
0·01	0·99995	15·1040	56·466	40·814
0	1·00000	15·1038	56·465	40·812

TABLE 1. Pressure drop resistance coefficients. The bipolar co-ordinate parameter α is defined as $a/h = \operatorname{sech}\alpha$

$0 < a/h \leqslant 1$. These values were computed from exact solutions of the Stokes equations in spherical bipolar co-ordinates, and from asymptotic singular perturbation analyses. The notation of these authors is related to that of the present text by the relations

$$K_0^r = -F_x^{r*}, \quad L_0^r = -T_y^{r*}, \tag{3.30a, b}$$

$$K_0^s = F_x^{s*}, \quad L_0^s = \tfrac{1}{2}(a/h)\,T_y^{s*}, \tag{3.31a, b}$$

$$K_0^t = -F_x^{t*}, \quad L_0^t = -T_y^{t*}. \tag{3.32a, b}$$

Values of K_0^t and L_0^t had previously been calculated by O'Neill (1964), who also computed K_0^s and L_0^s for $a/h = 1$ (O'Neill 1968).

Values of the pressure drop coefficients P_2^r and P_2^t presented in table 1 were calculated from the results of Goldman *et al.* (1967*b*) by employing (3.31) in conjunction with the identities (3.28) and (3.29).† Computation of values of the remaining coefficient P_2^s appearing in table 1 required use of the following integral expression, obtained from (3.25) and (3.26):

$$P_2^s = \frac{4}{\pi}\left(\frac{h}{a}\right)\int_{S_p} d\hat{\mathbf{S}}\,.\,\boldsymbol{\pi}_0^s\,.\,\mathbf{u}_0. \tag{3.33}$$

The dimensionless pressure tensor $\boldsymbol{\pi}_0^s$ appearing in the above integrand is that arising from the solution $(\mathbf{v}_0, p_0)$ of (3.7)–(3.10) for a stationary sphere,

$$\hat{U}_0 = \hat{\omega} = 0.$$

† The asymptotic forms of the pressure drop coefficients as $a/h \to 0$ are as follows:

$$P_2^r = 16\left[1-\frac{3}{16}\left(\frac{a}{h}\right)^3\right]+O\left(\frac{a}{h}\right)^4,$$

$$P_2^t = 24\left(\frac{h}{a}\right)\left[1-\frac{5}{16}\left(\frac{a}{h}\right)^3\right]\left[1-\frac{9}{16}\left(\frac{a}{h}\right)+\frac{1}{8}\left(\frac{a}{h}\right)^3-\frac{45}{256}\left(\frac{a}{h}\right)^4\right]^{-1}+O\left(\frac{a}{h}\right)^4,$$

$$P_2^s = 24\left(\frac{h}{a}\right)\left[1+\frac{8}{9}\left(\frac{a}{h}\right)^2-\frac{5}{8}\left(\frac{a}{h}\right)^3\right]\left[1-\frac{9}{16}\left(\frac{a}{h}\right)+\frac{1}{8}\left(\frac{a}{h}\right)^3\right]^{-1}+O\left(\frac{a}{h}\right)^2.$$

These reflexion-type expansions were obtained from the comparable reflexion-type expressions for the force and torque resistance coefficients summarized by Goldman *et al.* (1967*a*, *b*), the analysis of Wakiya, Darabaner & Mason (1967), and the limiting form of the additional pressure drop for a neutrally buoyant sphere suspended in a Poiseuille flow (Brenner 1970), $\Delta P_2^+ = \frac{40}{3}(a/h)+O(a/h)^2$, derived from (3.24), (3.17) and (4.14) in the limit where $\beta \to 1$.

The exact bipolar co-ordinate solution of this problem (Goren & O'Neill 1971) was used to effect an analytic integration of (3.34), using techniques developed by Goldman (1966) for the evaluation of a similar integral (Goldman *et al.* 1967*b*).†

4. Results and discussion

The asymptotic expressions of § 3 will be applied in two specific situations of importance: the sedimentation of a homogeneous non-neutrally buoyant sphere in an otherwise quiescent fluid; and the motion of a neutrally buoyant sphere suspended in a Poiseuille flow.

Sedimentation of a homogeneous sphere in a quiescent fluid ($V_m = 0$)

A sphere settling at its terminal speed is acted upon by a hydrodynamic force which is equal in magnitude, but opposite in direction, to the force of gravity **g** (corrected for the buoyant effect of the fluid), $F = \frac{4}{3}\pi a^3(\rho_s - \rho)\, g$. As the sphere is homogeneous, no external gravitational couple acts on it (i.e., $T_o = 0$), whence the sphere is freely rotating. According to (2.13) the angular velocity of such a sphere is related to its terminal translational velocity by the expression

$$\omega = -(L^t/L^r)\, U_o. \tag{4.1}$$

In the absence of the constraining effect of the tube walls the sphere translates without rotation, its sedimentation velocity being given by Stokes' law as

$$U_\infty = -F/6\pi\mu a. \tag{4.2}$$

On substituting (4.1) and (4.2) into (2.13) one finds, in comparison with the unbounded case, that the sphere in the tube settles at a velocity U_o given by the relation

$$U_o/U_\infty = 6\pi a[K^t - (L^t/L^r)\, K^r]^{-1}. \tag{4.3}$$

Sedimentation creates a pressure differential ΔP^+ in the fluid (Feldman & Brenner 1968), the pressure in the fluid being highest on that side of the sphere towards which it translates. From (2.13) and (4.1) this pressure diminution is

$$\Delta P^+ A/\mu U_o = (L^t/L^r)\, P^r - P^t.$$

Alternatively, use of (2.13) in conjunction with the reciprocity conditions (2.15)–(2.17) gives the equivalent formula

$$\frac{\Delta P^+ A}{F} = \frac{K^s L^r - K^r L^s}{K^t L^r - K^r L^t}. \tag{4.4}$$

Relations (4.3) and (4.4) apply for a sphere of any size or position in the tube. For a small sphere ($\lambda \ll 1$) they adopt the asymptotic (dimensionless) forms

$$U_o/U_\infty = [K_0^t - (L_0^t/L_0^r)\, K_0^r]^{-1} + O(\lambda) \tag{4.5}$$

and

$$\frac{\Delta P^+ A}{F} = 4\left(\frac{h}{a}\right) \frac{K_0^s L_0^r - K_0^r L_0^s}{K_0^t L_0^r - K_0^r L_0^t}\, \lambda + O(\lambda^2). \tag{4.6}$$

These forms are valid for expansions of the type discussed in § 3 ($\lambda \ll a/h \leqslant 1$), as well as for 'reflexion-type' expansions ($\lambda \leqslant a/h \ll 1$).

† See footnote on previous page.

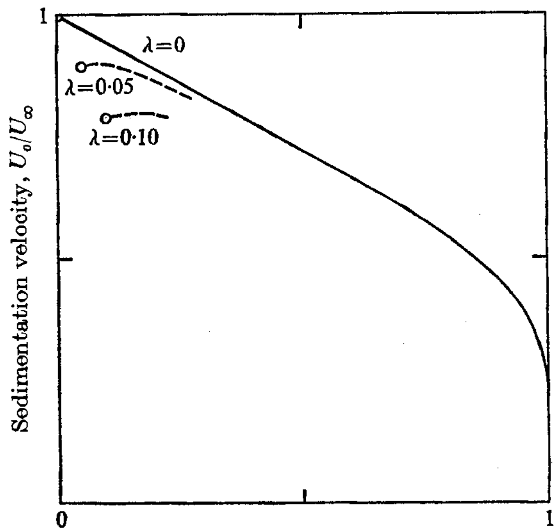

FIGURE 2. Terminal sedimentation velocity of a sphere settling freely in a vertical tube ($T_o = 0$, $V_m = 0$): ———, perturbation expansion (4.5); – – – –, method of reflexions expansion (4.7); O, sphere centre on tube axis.

The effect of the wall on the sedimentation velocity of a small sphere is illustrated in figure 2. The dashed lines are obtained from the method of reflexions expansion (Brenner 1966)

$$U_o/U_\infty = 1-f(\beta)\,\lambda+O(\lambda^3) \quad (\beta \leqslant 1-\lambda), \tag{4.7}$$

in which the argument β of the wall-effect function f represents the fractional radial distance of the sphere centre from the tube axis:

$$\beta = 1-(h/a)\,\lambda. \tag{4.8}$$

Numerical tabulations of the function $f(\beta)$ *vs.* β in the range $0 \leqslant \beta \leqslant 1$ are furnished by Greenstein & Happel (1968) and, more completely, by Hirschfeld (1972). The values of $f(\beta)$ decrease slightly from $f(0) = 2{\cdot}10444$ to a minimum at $\beta \approx 0{\cdot}41$ before approaching the asymptote $f \sim (9/16)(1-\beta)^{-1}$ as $\beta \to 1$. The circled points from which the curves originate denote the velocity ratios for sedimentation along the tube axis. The solid line in figure 2 is a plot of the first term of expansion (4.5), obtained using values of the resistance coefficients tabulated by Goldman *et al.* (1967*a*). The solid and dashed lines will tend to agree for the intermediate parameter range $\lambda \ll a/h \ll 1$ common to both expansion procedures.

Not surprisingly, the velocity ratio U_o/U_∞ never exceeds unity, the influence of the cylindrical boundary being to retard the settling motion of the sphere at all radial positions. Provided the constraint $\lambda \ll 1-\beta$ is satisfied the settling velocity for a given λ is greatest at the lateral position ($\beta \approx 0{\cdot}41$) corresponding to the minimum in $f(\beta)$. For $\beta > 0{\cdot}41$, the closer the sphere is to the wall the more slowly it settles. In fact, according to the solid curve in figure 2, the settling

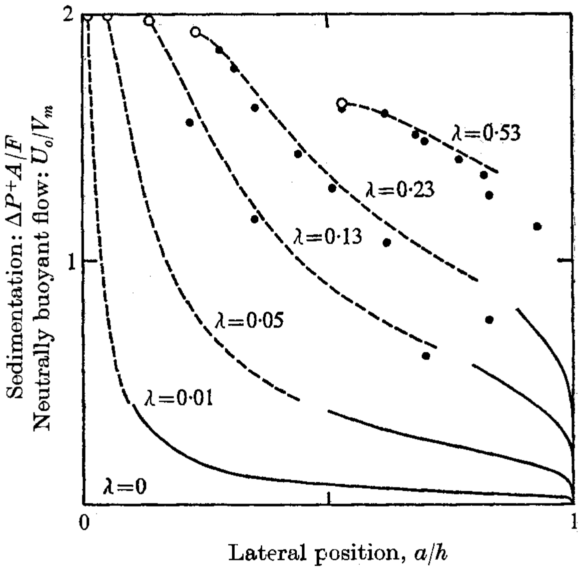

FIGURE 3. Pressure drop created by sphere settling in a vertical tube ($T_0 = 0$, $V_m = 0$) or translational velocity of a neutrally buoyant sphere ($F = 0$, $T_0 = 0$): ———, perturbation expansion (4.6); – – – –, method of reflexions expansion (4.9); ○, sphere centre on tube axis; ●, data of Goldsmith & Mason (1962).

velocity decreases rapidly to zero as $a/h \to 1$. Theoretically, contact with the wall would bring the particle to rest ($U_0 = 0, \omega = 0$), since an infinite force would then be required to produce motion of the sphere (Goldman *et al.* 1967*a*,*b*; O'Neill & Stewartson 1967). As was discussed by Goldman *et al.* (1967*a*) such behaviour is unlikely to be observed in practice owing to a breakdown in the assumptions regarding perfect smoothness of the rigid surfaces and the integrity and constancy of physical properties of the fluid phase.

According to the method of reflexions, the pressure difference accompanying this particle sedimentation is (Brenner 1966)

$$\Delta P^{+}A/F = 2(1-\beta^2)-\tfrac{4}{3}\lambda^2+O(\lambda^3) \quad (\beta \ll 1-\lambda). \tag{4.9}$$

The complementary perturbation expansion was evaluated by substituting into (4.6) the appropriate resistance coefficients tabulated by Goldman *et al.* (1967*b*). Both results are plotted in figure 3. As with the settling velocity, the pressure drop is a function of both λ and a/h. Comparison with figure 2 suggests that the variation of pressure drop with lateral position is only partly attributable to the change of settling velocity with position. For a given sphere and tube the maximum pressure drop occurs for settling along the tube axis, $\beta = 0$.

Neutrally buoyant sphere in a Poiseuille flow

In the absence of external forces and torques the sphere is freely suspended by the fluid ($F = 0, T_0 = 0$). Solution of (2.13) for this case yields the following values for the translational and angular velocities of the neutrally buoyant sphere:

$$\frac{U_0}{V_m} = \frac{K^s L^r - K^r L^s}{K^t L^r - K^r L^t}, \quad \frac{\omega}{V_m} = \frac{K^s L^t - K^t L^s}{K^t L^r - K^r L^t}. \tag{4.10), (4.11}$$

From (2.13), the additional pressure drop resulting from the presence of such a sphere is then

$$\Delta P^{+}A/\mu V_m = P^s - (U_o/V_m)\, P^t - (\omega/V_m)\, P^r, \tag{4.12}$$

in which the appropriate expressions for the velocity ratios are those given by (4.10) and (4.11).

It is interesting to observe that the right-hand side of (4.10) is identical to that of (4.4). Indeed, the equivalence of these two equations can be demonstrated directly via use of the reciprocal theorem of §2:

$$(\Delta P^{+}A/F)_{V_m=0,\, T_o=0} = (U_o/V_m)_{F=0,\, T_o=0}. \tag{4.13}$$

Hence, the asymptotic forms adopted by the neutrally buoyant translational velocity of (4.10) can be obtained from (4.6), (4.9) and (4.13). The remarks made above concerning figure 3 are now seen to apply to this velocity ratio U_o/V_m too. Conversely, the dependence of the pressure drop upon the size and lateral position of a sedimenting sphere in a quiescent fluid can be determined experimentally from measurement of the transitional velocity of an identical, but neutrally buoyant, sphere at the same lateral position. Such information, albeit for relatively large spheres, is available from the experimental measurements of Goldsmith & Mason (1962) of the particle velocities in a dilute suspension of neutrally buoyant spheres suspended in a Poiseuille flow. Their data for the translational velocities are replotted in figure 3 as the black circles. It would not be normally expected that the truncated asymptotic expansions for $\lambda \ll 1$ would produce accurate predictions in the upper portion of the range, $0{\cdot}13 < \lambda < 0{\cdot}53$, covered by these experiments. Nevertheless, reasonable agreement exists between the data and the method of reflexions expansion.

In figure 4 representative plots are presented for the dominant contributions to the additional pressure drop engendered by a small neutrally buoyant sphere. The dashed lines – valid when the sphere is far from the wall – were calculated from the expansion (Brenner 1966, 1970)

$$\Delta P^{+}R_o/\mu V_m = \tfrac{160}{3}\beta^2\lambda^3 + O(\lambda^5) \quad (\beta \ll 1-\lambda). \tag{4.14}$$

For a given λ the minimum pressure drop arises when the sphere moves along the tube axis, $\beta = 0$. Although the upper bound of the eccentricity parameter is $\beta = 1$, the maximum value of the contribution of order λ^3 is not $53{\cdot}3\lambda^3$, as would be obtained by letting $\beta \to 1$ in (4.14). Rather, as can be seen from figure 4, the perturbation expansion for the case of a sphere close to the wall predicts the maximum value to be $226\lambda^3$ – a value over four times as large as that predicted by the method of reflexions. This value is arrived at by expressing (4.12) as the expansion

$$\Delta P^{+}R_o/\mu V_m = 4(h/a)\,[P_2^s - (U_o/Gh)\, P_2^t - (\omega a/Gh)\, P_2^r]\,\lambda^3 + O(\lambda^4), \tag{4.15}$$

in which $G = 4V_m/R_o$ is the Poiseuille shear rate at the tube wall. Numerical values for the coefficient of λ^3 in (4.15) were computed from the pressure drop coefficients of table 1 and the neutrally buoyant velocity ratio values of Goldman *et al.* (1967*b*). The maximum, occurring at $a/h = 1$, corresponds to the

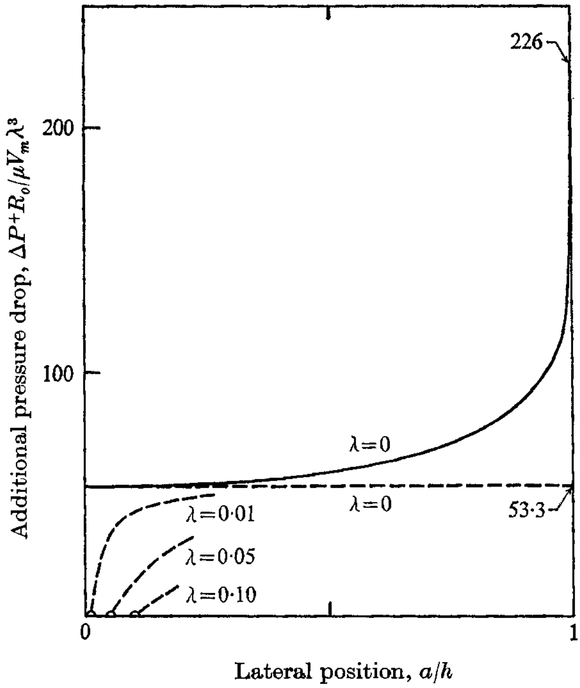

FIGURE 4. Additional pressure drop created by a neutrally buoyant sphere in Poiseuille flow ($F = 0$, $T_0 = 0$): ———, perturbation expansion (4.15); ----, method of reflexions expansion (4.14); ○, sphere centre on tube axis.

additional pressure drop for a stationary sphere ($U_0 = \omega = 0$) in contact with the tube wall.

It has been suggested by Greenstein & Happel (1970) that the method of reflexions solution for a single sphere in a tube can be used to predict the pressure drop–flow rate relationship for the tube flow of dilute suspensions of non-hydrodynamically interacting spheres. These authors obtained an explicit asymptotic result, assuming the distribution of spheres to be uniform across the tube. The contributions (4.14) of each of the individual spheres were 'summed', by application of the equation

$$(\Delta P^{+}A)_{\text{total}} = 2\pi \int_{\beta=0}^{1} \Delta P^{+}(\beta)\,\beta\, d\beta. \tag{4.16}$$

Though this procedure may produce an adequate approximation for very small λ, the present work indicates that it is fundamentally improper to use the method of reflexions expansion over the entire tube cross-section. Equation (4.16) ignores the contribution of the complementary expansion (4.15) for spheres located immediately adjacent to the tube wall.

The symmetry of the resistance matrix in (2.13) is closely related to the corresponding symmetry of a similar (partitioned) matrix recently introduced by Hinch (1972) for an isolated particle in an unbounded fluid subjected to a linear shear flow. Inclusion of a solid plane boundary on whose surface $\mathbf{v} = \mathbf{0}$ does not affect the validity of Hinch's (1972) arguments. In place of the pressure drop force $\Delta P^{+}A$ in our analysis there appears a 'stresslet' **S**, defined as

$$\mathbf{S} = \frac{1}{2}\int_{S_p} (\mathbf{r}\, d\mathbf{S}\,.\,\boldsymbol{\pi} + d\mathbf{S}\,.\,\boldsymbol{\pi}\mathbf{r}). \tag{4.17}$$

In view of (3.22) and (3.13)–(3.14), a close correspondence clearly exists between the stresslet and the additional pressure drop force for a force-free particle,

$$\int_{S_p} d\mathbf{S}\,.\,\boldsymbol{\pi} = 0.$$

This equivalence is not surprising in view of the fact that the stresslet and the term $\Delta P^{+}AV_m$ are both related to the additional energy dissipation in a linear shear flow.

P.M.B. was supported by Public Health Service Postdoctoral Fellowship no. FO2HE43778 administered by the National Heart and Lung Institute, and a NATO Postdoctoral Fellowship from the National Science Foundation. H.B. was supported by the National Science Foundation under Grant no. GK-34855.

REFERENCES

Brenner, H. 1964 The Stokes resistance of an arbitrary particle. II. An extension. *Chem. Engng Sci.* **19**, 599–629.

Brenner, H. 1966 Hydrodynamic resistance of particles at small Reynolds numbers. In *Advances in Chemical Engineering*, vol. VI (ed. T. B. Drew, J. W. Hoopes & T.Vermuelen), pp. 287–438. Academic.

Brenner, H. 1970 Pressure drop due to the motion of neutrally buoyant particles in duct flows. *J. Fluid Mech.* **43**, 641–660.

Brenner, H. & Happel, J. 1958 Slow viscous flow past a sphere in a cylindrical tube. *J. Fluid Mech.* **4**, 195–213.

Bungay, P. M. 1970 Ph.D. thesis, Carnegie–Mellon University, Pittsburgh.

Bungay, P. M. & Brenner, H. 1973*a* Pressure drop due to the motion of neutrally buoyant particles in duct flows. III. Non-neutrally buoyant spherical droplets and bubbles. *Z. angew. Math. Mech.* **53**, 187–192.

Bungay, P. M. & Brenner, H. 1973*b* The motion of a closely-fitting sphere through a fluid-filled tube. *Int. J. Multiphase Flow*, **1** (in press).

Cox, R. G. & Brenner, H. 1967 Effect of finite boundaries on the Stokes resistance of an arbitrary particle. Part 3. Translation and rotation. *J. Fluid Mech.* **28**, 391–411.

Cox, R. G. & Mason, S. G. 1971 Suspended particles in fluid flow through tubes. *Ann. Rev. Fluid Mech.* **3**, 291–316.

Feldman, G. A. & Brenner, H. 1968 Experiments on the pressure drop created by a sphere settling in a viscous liquid. Part 2. Reynolds numbers from 0·2 to 21000. *J. Fluid Mech.* **32**, 705–720.

Goldman, A. J. 1966 Investigations in low Reynolds number fluid-particle dynamics. Ph.D. thesis, New York University.

Goldman, A. J., Cox, R. G. & Brenner, H. 1967*a* Slow viscous motion of a sphere parallel to a plane wall. I. Motion through a quiescent fluid. *Chem. Engng Sci.* **22**, 637–651

Goldman, A. J., Cox, R. G. & Brenner, H. 1967*b* Slow viscous motion of a sphere parallel to a plane wall. II. Couette flow. *Chem. Engng Sci.* **22**, 653–660.

Goldsmith, H. L. & Mason, S. G. 1962 The flow of suspensions through tubes. I. Single spheres, rods, and discs. *J. Colloid Sci.* **17**, 448–476.

Goren, S. L. & O'Neill, M. E. 1971 On the hydrodynamic resistance to a particle of a dilute suspension when in the neighbourhood of a large obstacle. *Chem. Engng Sci.* **26**, 325–338.

Greenstein, T. & Happel, J. 1968 Theoretical study of the slow motion of a sphere and a fluid in a cylindrical tube. *J. Fluid Mech.* **34**, 705–710.

Greenstein, T. & Happel, J. 1970 Viscosity of dilute uniform suspensions of uniform spheres. *Phys. Fluids*, **13**, 18–21.

Haberman, W. L. & Sayre, R. M. 1958 Motion of rigid and fluid spheres in stationary and moving liquids inside cylindrical tubes. *David Taylor Model Basin Rep.* no. 1143.

Hinch, E. J. 1972 Note on the symmetries of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.* **54**, 423–425.

Hirschfeld, B. R. 1972 A theoretical study of the slow, asymmetric settling motion of an arbitrarily positioned particle in a circular cylinder. Ph.D. thesis, New York University.

Hochmuth, R. M. & Sutera, S. P. 1970 Spherical caps in low Reynolds number tube flow. *Chem. Engng Sci.* **25**, 593–604.

O'Neill, M. E. 1964 A slow motion of viscous liquid caused by a slowly moving solid sphere. *Mathematika*, **11**, 67–74.

O'Neill, M. E. 1968 A sphere in contact with a plane wall in a slow linear shear flow. *Chem. Engng Sci.* **23**, 1293–1298.

O'Neill, M. E. & Stewartson, K. 1967 On the slow motion of a sphere parallel to a nearby plane wall. *J. Fluid Mech.* **27**, 705–724.

Sonshine, R. M., Cox, R. G. & Brenner, H. 1966 The Stokes translation of a particle of arbitrary shape along the axis of a circular cylinder filled to a finite depth with viscous liquid. *Appl. Sci. Res.* **16**, 273–300.

Wakiya, S., Darabaner, C. L. & Mason, S. G. 1967 Particle motions in sheared suspensions. XXI. Interactions of rigid spheres (theoretical). *Rheol. Acta*, **6**, 264–273.

Wang, H. & Skalak, R. 1969 Viscous flow in a cylindrical tube containing a line of spherical particles. *J. Fluid Mech.* **38**, 75–96.